Topology-induced coarsening in language games

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We investigate how very large populations are able to reach a global consensus, out of local "microscopic" interaction rules, in the framework of a recently introduced class of models of semiotic dynamics, the so-called naming game. We compare in particular the convergence mechanism for interacting agents embedded in a low-dimensional lattice with respect to the mean-field case. We highlight that in low dimensions consensus is reached through a coarsening process that requires less cognitive effort of the agents, with respect to the mean-field case, but takes longer to complete. In one dimension, the dynamics of the boundaries is mapped onto a truncated Markov process from which we analytically computed the diffusion coefficient. More generally we show that the convergence process requires a memory per agent scaling as N and lasts a time $N^{1+2/d}$ in dimension $d \leq 4$ (the upper critical dimension), while in mean field both memory and time scale as $N^{3/2}$, for a population of N agents. We present analytical and numerical evidence supporting this picture.

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The past decade has seen an important development of the so-called semiotic dynamics, a field that studies how conventions (or semiotic relations) can originate, spread, and evolve over time in populations. This occurred mainly through the definition of language interaction games [1,2] in which a population of agents is seen as a complex adaptive system that self-organizes [3] as a result of simple local interactions (games). The interest of physicists for language games comes from the fact that they can be easily formulated as nonequilibrium statistical mechanics models of interacting agents: At each time step, an agent updates its state (among a certain set of possible states) through an interaction with its neighbors. An interesting question concerns the possibility of convergence towards a common state for all agents, which emerges without external global coordination and from purely local interaction rules [4–6]. In this Rapid Communication, we focus on the so-called naming games, introduced to describe the emergence of conventions and shared lexicons in a population of individuals interacting with each other by negotiations rules, and study how the embedding of the agents on a low-dimensional lattice influences the emergence of consensus, which we show to be reached through a coarsening process. The original model [7] was inspired by a well-known experiment of artificial intelligence called Talking Heads [8], in which embodied software agents develop their vocabulary observing objects through digital cameras, assigning them randomly chosen names, and negotiating these names with other agents.

Recently a minimal version of the naming game endowed with simplified interactions rules [9] was introduced, which reproduces the phenomenology of the experiments and is amenable to analytical treatment. In this model, N individuals (or agents) observe the same object, trying to communicate its name to one another. The identical agents have at their disposal an internal inventory, in which they can store an unlimited number of different names or opinions. At the initial time, all individuals have empty inventories, with no innate terms. At each time step, the dynamics consists of a pairwise interaction between randomly chosen individuals. Each agent can take part in the interaction as a "speaker" or as a "hearer." The speaker transmits to the hearer a possible name for the object at issue; if the speaker does not know a name for the object (its inventory is empty), it invents a name to be passed to the hearer [10]. In the case where it already knows more synonyms (stored in the inventory), it chooses one of them randomly. The hearer's move is deterministic: If it possesses the term pronounced by the speaker, the interaction is a success, and both speaker and hearer retain that name as the right one, canceling all the other terms in their inventories; otherwise, the new name is included in the inventory of the hearer, without any cancellation.

The mean-field (MF) case has been studied in [9]: The system initially accumulates a large number of possible names for the object since different agents (speakers) initially invent different names and propagate them. Interestingly, however, this profusion of different names leads in the end to an asymptotic absorbing state in which all the agents share the same name.

Although this model leads to the convergence of all agents to a common state or "opinion," it is interesting to note the important differences with other commonly studied models of opinion formation [4–6]. For example, each agent can potentially be in an infinite number of possible discrete states (or words, names), contrary to the Voter model in which each agent has only two possible states [6]. Moreover, an agent can here accumulate in its memory different possible names for the object, i.e., wait before reaching a decision. Finally, each dynamical step involves a certain degree of stochasticity, while in the Voter model, an agent deterministically adopts the opinion of one of its neighbors.

In this paper, we study the naming game model on low-dimensional lattices: the agents, placed on a regular d-dimensional lattice, can interact only with their $2d$ nearest neighbors. Numerical and analytical investigations allow us to highlight important differences with the mean-field case, in particular in the time needed to reach consensus, and in the effective size of the inventories or total memory required. We show how the dynamics corresponds to a coarsening of clusters of agents sharing a common name; the interfaces between such clusters are composed by agents who still have more than one possible name.
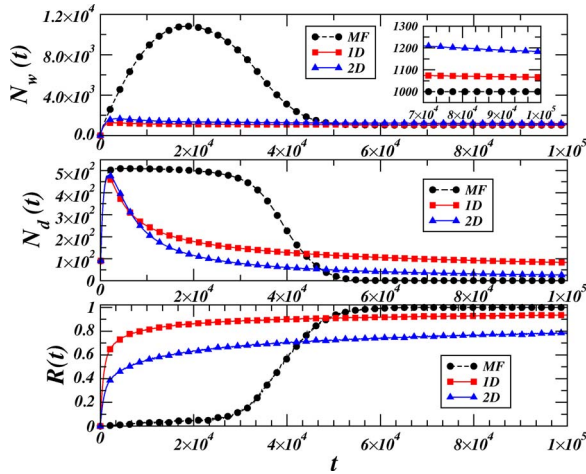
FIG. 1. (Color online) Time evolution in mean-field and finite dimensions of the total number of words (or total used memory) for the number of different words in the system, and for the average success rate. $N=1024$, average over 1000 realizations. The inset in the top graph shows the very slow convergence in finite dimensions.

Relevant quantities in the study of naming games are the total number of words in the system $N_w(t)$, which corresponds to the total memory used by the agents, the total number of different words $N_d(t)$, and the average rate of success $S(t)$ of the interactions. Figure 1 displays the evolution in time of these three quantities for the low-dimensional models, compared to the mean-field case.

In the initial state, all inventories are empty. At short times, therefore, each speaker with an empty inventory has to invent a name for the object, and many different words are indeed invented. In this initial phase, the success rate is equal to the probability that two agents that have already played are chosen again: This rate is proportional to t/E where E is the number of possible interacting pairs, i.e., $N(N-1)/2$ for the mean-field case and Nd in finite dimensions. $S(t)$ grows thus N times faster in finite dimensions, as confirmed by numerics. At larger times, however, the eventual convergence is much slower in finite dimensions.

The curves for $N_w(t)$ and $N_d(t)$ display in all cases a sharp increase at short times, a maximum for a given time t_{max} and then a decay towards the consensus state in which all the agents share the same unique word, reached at t_{conv}. The short time regime corresponds to the creation of many different words by the agents. After a time of order N, each agent has played typically once, and therefore $O(N)$ different words have been invented (in fact, typically $N/2$): The total number of *distinct* words in the system grows and reaches a maximum scaling as N. Due to the interactions, the agents accumulate in memory the words they have invented and the words that other agents have invented. In MF, each agent can interact with all the others, so that it can learn many different words, and in fact the maximal memory necessary for each agent scales as $N^{\alpha_{MF}}$ with $\alpha_{MF}=0.5$ [9], so that the total memory used at the peak is $\sim N^{1.5}$, with many words shared by many agents, and $t_{max} \sim N^{\beta_{MF}}$ with $\beta_{MF}=1.5$. Moreover, during this learning phase, words are not eliminated [$S(t)$ is very small] so that the total number of distinct words displays a long plateau. The redundancy of words then reaches
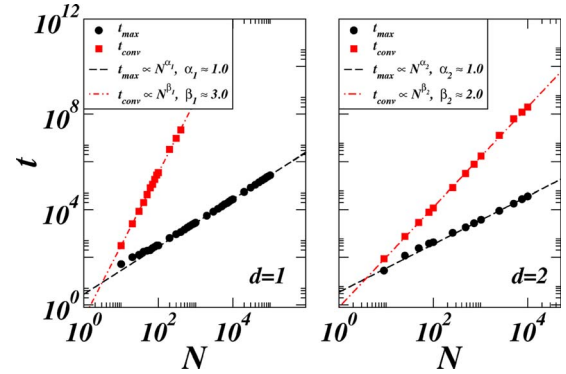

FIG. 2. (Color online) Scaling of the time at which the number of words is maximal, and of the time needed to obtain convergence, in one and two dimensions.

a sufficient level to begin producing successful interactions and the decrease of the number of words is then very fast, with a rapid convergence to the consensus state. In contrast, in finite dimensions words can only spread locally, and each agent has access only to a finite number of different words. The total memory used scales as N, and the time t_{max} to reach the maximum number of words in the system scales as N^{α_d} with $\alpha_1=\alpha_2=1$ (Fig. 2). No plateau is observed in the total number of distinct words since the coarsening of clusters of agents soon starts to eliminate words.

Furthermore, the time needed to reach consensus, t_{conv}, grows as N^{β_d} with $\beta_1 \simeq 3$ in $d=1$ and $\beta_2 \simeq 2$ in $d=2$, while $\beta_{MF} \simeq 1.5$ (Fig. 2). We will now see how such behaviors emerge from a more detailed numerical and analytical analysis of the dynamical evolution.

Figure 3 reports a typical evolution of agents on a one-dimensional lattice, by displaying one below the other a certain number of (linear) configurations corresponding to successive equally separated temporal steps. Each agent having one single word in memory is presented by a colored point while agents having more than one word in memory are shown in black. This figure clearly shows the growth of clusters of agents having one single word by diffusion of inter-
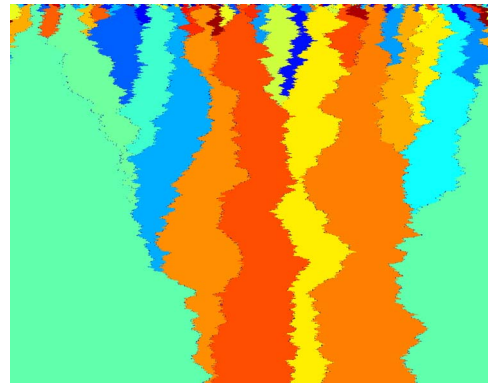

FIG. 3. (Color online) Typical evolution of a one-dimensional system ($N=1000$). Black corresponds to interfaces (sites with more than one word). The other colors identify different single state clusters. The vertical axis represents the time ($1000 \times N$ sequential steps); the one-dimensional snapshots are reported on the horizontal axis.
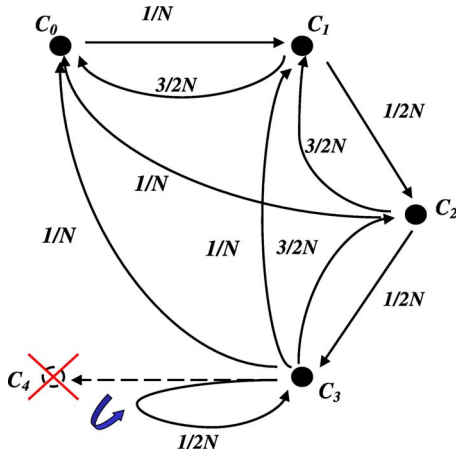
FIG. 4. (Color online) Truncated Markov process associated with interface width dynamics-schematic evolution of a C_0 interface $\cdots AAABBB\cdots$, cut at the maximal width $m=3$.

faces made of agents having more than one word in memory. The fact that the interfaces remain thin is, however, not obvious *a priori*: An agent having, e.g., two words in memory can propagate them to its neighbors, leading to possible clusters of agents having more than one word.

In order to rationalize and quantify such evolution, we consider a single interface between two linear clusters of agents: In each cluster, all the agents share the same unique word, say A in the left-hand cluster and B in the other. The interface is a string of length m composed of sites in which both states A and B are present. We call C_m this state $(A+B)^m$. A C_0 corresponds to two directly neighboring clusters ($\cdots AAABBB\cdots$), while C_m means that the interface is composed by m sites in the state $C=A+B$ ($\cdots AAAC\cdots CBBB\cdots$). Note that, in the actual dynamics, two clusters of states A and B can be separated by a more complex interface. For instance a C_m interface can break down into two or more smaller sets of C states spaced out by A or B clusters, causing the number of interfaces to grow. Numerical investigation shows that such configurations are, however, eliminated in the early times of the dynamics.

Bearing in mind these hypotheses, an approximate expression for the stationary probability that two neighboring clusters are separated by a C_m interface can be computed in the following way. In a one-dimensional line composed of N sites and initially divided into two clusters of A and B, the probability to select the unique C_0 interface is $1/N$, and the interacting rules say that the only possible product is a C_1 interface. Thus there is a probability $p_{0,1}=1/N$ that a C_0 interface becomes a C_1 interface in a single time step; otherwise it stays in C_0. From C_1 the interface can evolve into a C_0 or a C_2 interface with probabilities $p_{1,0}=3/2N$ and $p_{1,2}=1/2N$, respectively. This procedure is easily extended to higher values of m. The numerics suggest that we can safely truncate this study at $m \leqslant 3$. In this approximation, the problem corresponds to determining the stationary probabilities of the Markov chain reported in Fig. 4 and defined by transition matrix

$$
\mathcal{M} = \begin{pmatrix}
\frac{N-1}{N} & \frac{1}{N} & 0 & 0 \\
\frac{3}{2N} & \frac{N-2}{N} & \frac{1}{2N} & 0 \\
\frac{1}{N} & \frac{3}{2N} & \frac{N-3}{N} & \frac{1}{2N} \\
\frac{1}{N} & \frac{1}{N} & \frac{3}{2N} & \frac{N-4}{N}+\frac{1}{2N}
\end{pmatrix}, \quad (1)
$$

in which the basis is $\{C_0, C_1, C_2, C_3\}$ and the contribution $1/2N$ from C_3 to C_4 has been neglected (see Fig. 4). The stationary probability vector $\mathbf{P}=\{P_0, P_1, P_2, P_3\}$ is computed by imposing $\mathbf{P}(t+1)-\mathbf{P}(t)=0$, i.e., $(\mathcal{M}^T-I)\mathbf{P}=0$, which gives $P_0=133/227 \approx 0.586$, $P_1=78/227 \approx 0.344$, $P_2=14/227 \approx 0.062$, $P_3=2/227 \approx 0.0088$. Direct numerical simulations of the evolution of a line $\cdots AAABBB\cdots$ yields $P_0 \simeq 0.581$, $P_1=0.344$, $P_2=0.063$, $P_3=0.01$, thus clearly confirming the correctness of our approximation.

Since our analysis shows that the width of the interfaces remains small, we assume that they are punctual objects localized around their central position x: In the previously analyzed case, denoting by x_l the position of the rightmost site of cluster A and by x_r the position of the leftmost site of cluster B, it is given by $x=(x_l+x_r)/2$. An interaction involving sites of an interface, i.e., an interface transition $C_m \to C_{m'}$, corresponds to a set of possible movements for the central position x. The set of transition rates are obtained by enumeration of all possible cases: Denoting by $W(x \to x \pm \delta)$ the transition probability that an interface centered in x moves to the position $x \pm \delta$, in our approximation only three symmetric contributions are present. We obtain

$$
W\left(x \to x \pm \frac{1}{2}\right) = \frac{1}{2N}P_0 + \frac{1}{N}P_1 + \frac{1}{N}P_2 + \frac{1}{2N}P_3,
$$

$$
W(x \to x \pm 1) = \frac{1}{2N}P_2 + \frac{1}{2N}P_3,
$$

$$
W\left(x \to x \pm \frac{3}{2}\right) = \frac{1}{2N}P_3.
$$

Using the expressions for the stationary probability $P_0, \ldots, P_3$, we finally get $W(x \to x \pm 1/2)=319/454N$, $W(x \to x \pm 1)=8/227N$, and $W(x \to x \pm 3/2)=1/227N$.

The knowledge of these transition probabilities allows us to write the master equation for the probability $\mathcal{P}(x,t)$ to find the interface in position x at time t, which, in the limit of continuous time and space (i.e. writing

$$
\mathcal{P}(x,t+1) - \mathcal{P}(x,t) \approx \delta t \frac{\partial \mathcal{P}}{\partial t}(x,t),
$$

while

$$
\mathcal{P}(x+\delta x,t) \approx \mathcal{P}(x,t) + \delta x \frac{\partial \mathcal{P}}{\partial x}(x,t) + \frac{(\delta x)^2}{2} \frac{\partial^2 \mathcal{P}}{\partial x^2}(x,t),
$$

reads

$$
\frac{\partial \mathcal{P}(x,t)}{\partial t} = \frac{D}{N} \frac{\partial^2 \mathcal{P}(x,t)}{\partial x^2},
$$

where $D=401/1816 \simeq 0.221$ is the diffusion coefficient [in the appropriate dimensional units $(\delta x)^2/\delta t$].
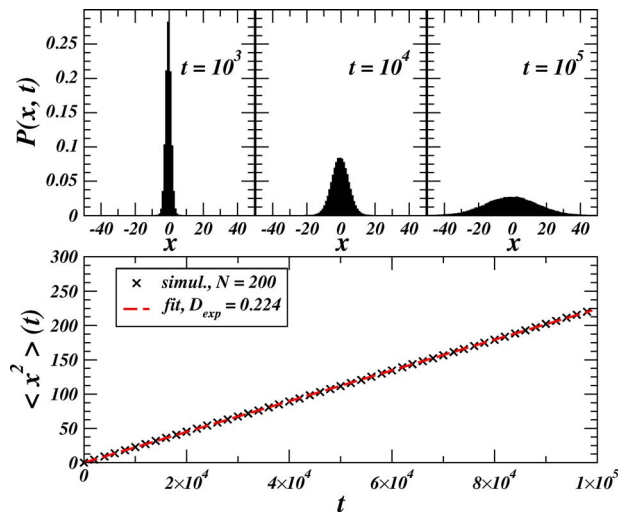
FIG. 5. (Color online) Evolution of the position of an interface $\cdots AAABBB\cdots$. Top, evolution of the distribution $\mathcal{P}(x,t)$. Bottom, evolution of the mean-square displacement, showing a clear diffusive behavior $\langle x^2\rangle = 2D_{exp}\,t/N$ with a coefficient $D_{exp}\approx 0.224$ in agreement with the theoretical prediction.

These results are confirmed by numerical simulations as illustrated in Fig. 5 where the numerical probability $\mathcal{P}(x,t)$ is shown to be a Gaussian around the initial position, while the mean-square distance reached by the interface at the time t follows the diffusion law $\langle x^2\rangle = 2D_{exp}\,t/N$ with $D_{exp}\simeq 0.224 \approx D$. The dynamical evolution of the naming game on a one-dimensional lattice can then be described as follows: At short times, pairwise interactions create $O(N)$ small clusters, divided by thin interfaces (see the first lines in Fig. 3). We can estimate the number of interfaces at this time with the number of different words in the lattice, which is about $N/2$. The interfaces then start diffusing. When two interfaces meet, the cluster situated in between the interfaces disappears, and the two interfaces coalesce. Such a coarsening leads to the well-known growth of the typical size ξ of the clusters as $t^{1/2}$. The density of interfaces, at which unsuccessful interactions can take place, decays as $1/\sqrt{t}$, so that $1-S(t)$ also decays as $1/\sqrt{t}$. Moreover, starting from a lattice in which all agents have no words, a time N is needed to reach a size of order 1,

so that in fact ξ grows as $\sqrt{t/N}$ (as also shown by the fact that the diffusion coefficient is D/N), which explains the time $t_{conv}\sim N^3$ needed to reach consensus, i.e., $\xi=N$.

This framework can be extended to the case of higher dimensions. The interfaces, although quite rough, are well defined and their width does not grow in time, which points to the existence of an effective surface tension. The numerical computation of equal-time pair correlation function in dimension $d=2$ (not shown) indicates that the characteristic length scale ξ grows as $\sqrt{t/N}$ [a time $O(N)$ is needed to initialize the agents to at least one word and therefore to reach a cluster size of order 1], in agreement with coarsening dynamics for nonconserved fields [11]. Since t_{conv} corresponds to the time needed to reach $\xi=N^{1/d}$, we can argue $t_{conv}\sim N^{1+2/d}$, which has been verified by numerical simulations in $d=2$ and $d=3$. This scaling and the observed coarsening behavior suggest that the upper critical dimension for this system is $d=4$ [11].

In conclusion, the study of the low-dimensional naming game using statistical physics methods provides a deeper understanding of the macroscopical collective dynamics of the model. We have shown how it presents a very different behavior in low-dimensional lattices than in mean field, indicating the existence of a finite upper-critical dimension. Low-dimensional dynamics is initially more effective; less memory per node is required, preventing agents from learning a large part of the many different words created. The dynamics then proceeds by the growth of clusters by coarsening, yielding a slow convergence to consensus. In contrast with other models of opinion dynamics (e.g., the Voter model [12,13]), the naming game presents an effective surface tension that is reminiscent of the nonequilibrium zero-temperature Ising model [11]. In this respect, it seems interesting to investigate the dynamics of the naming game in other topologies, such as complex networks in which each node has a finite number of neighbors combined with "long-range" links [14].

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